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SOME HYDRODYNAMIC ASPECTS OF SUPERFLUID HELIUM

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Abstract

A brief critical survey on the hydrodynamic formulations of the superfluid helium is given first. For the irreversible process, three major formulations, i.e. those due to Gorter-Mellink, Lin, and Hall-Vinen, Bekarevitch-Khalatnikov, are described, discussed and compared. Then some results of analyses based on the Gorter-Mellink formulation are presented. The paper concludes with some interesting findings resulting from the assumption that rotons are vortex rings.

Some Hydrodynamic Aspects of Superfluid Helium

I. Introduction.

The substance He^4 was first liquefied in 1908[1]. Its boiling point is 4.21°K, and it has a critical point with temperature 5.2°K and pressure 2.26 atm. Shortly after the liquefication of helium, it was found[2] that the liquid helium will undergo a phase transition at 2.18°K. This transition divides the liquid helium into two phases; the higher temperature phase is customarily called He I, while the lower temperature phase, He II. For the equilibrium property of the liquid, the phase transition is marked by, among others, a sharp maximum and a discontinuity of the slope in the density curve[3], and a logarithmic singularity in the specific heat curve[4], [5]. The shape of the specific curve leads to the terms "λ-transition" and "λ-point".

When the liquid is in motion, He I behaves like what one would expect any ordinary liquid at such low temperature. On the other hand, He II exhibits some very strange or "super" properties. We may mention:

(i) The ability to flow through extremely narrow channels[6] (superfluidity).

(ii) The normal damping of oscillations of immersed bodies[7].

(iii) The appearance of a fountain from a capillary when its other end is stuffed with powder and heated by radiation[8] (fountain effect).

(iv) The rise of temperature after the outflow through powder-plugged orifice[9], [10] (mechano-caloric effect).

(v) The propagation of temperature waves[11], [12], [13] (second sound).

(vi) The apparent loss of the super properties when the speed of liquid exceeds certain critical value[14], [15].

The peculiar properties of the superfluid helium are largely accountable on the basis of the phenomenological two-fluid theory[11], [12]. According to this theory, the liquid is considered to be a kind of mixture of two components, a normal component and a superfluid component. The density of the fluid ρ, can thus be separated into a normal density ρ_n and a superfluid density ρ_s

$$\rho = \rho_n + \rho_s \quad . \tag{1}$$

In the same manner, the fluid motion, characterized by its local velocity, $\underset{\sim}{v}$, may also be considered to be due to the concerted motions of the fluid components, so that

$$\rho \underset{\sim}{v} = \rho_n \underset{\sim}{v}_n + \rho_s \underset{\sim}{v}_s \quad , \tag{2}$$

where $\underset{\sim}{v}_n$ and $\underset{\sim}{v}_s$ are velocities of normal and superfluid components respectively. The normal component behaves just like any normal fluid, while the superfluid component is frictionless and carries no entropy.

The microscopic basis of the two-fluid theory of the liquid helium is somewhat different from the two-fluid theories of ordinary mixtures. Take an ordinary mixture, say, ionized gases; its components, say electrons and ionized atoms, are well defined physical particles whose detailed motion one can mentally or even actually follow. The components of He II, on the other hand, reveal themselves only collectively as a bulk fluid element. That we can not isolate in the classical sense the particles of normal and superfluid components is mainly due to the fact that He II is a fluid for which the quantum effects is important even in the macroscopic scale.

According to the prevailing physical theory[12], the normal component motion is a revelation of excitations in the superfluid "background". The superfluid background is the ground state liquid helium, hence is at zero temperature. They are not necessarily stationary in the macroscopic sense, for whether the bulk fluid is in motion or not is a relative matter. The motion of those macroscopic elements of "background" is revealed through $\underset{\sim}{v}_s$. The excitations in liquid helium is very much like the excitations in solids. In fact, here we also have phonons. In addition to phonons, we have "rotons" and other types of excitations. Based on this physical theory, we can obtain the macroscopic momenta and energy by the appropriate summation of the microscopic momenta and energies. However, the normal and superfluid densities are not basic, but derived quantities. Complete consistency is still lacking between the microscopic physical theory and the macroscopic hydrodynamic theory. In establishing the hydrodynamic theory of the liquid helium we still rely mainly on the arguments of continuum mechanics while incorporating it with the qualitative essentials of the microscopic theory.

II. Reversible Hydrodynamic Equations.

Based on the two-fluid theory, the following set of hydrodynamic equations may be derived[12], [16], [17].

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\epsilon \rho \underset{\sim}{v}_n + (1-\epsilon)\rho \underset{\sim}{v}_s] = 0 \quad , \tag{3}$$

$$\frac{\partial (\rho s)}{\partial t} + \nabla \cdot (\rho s \underset{\sim}{v}_n) = 0 \quad , \tag{4}$$

$$\frac{\partial \underset{\sim}{v}_n}{\partial t} + (\underset{\sim}{v}_n \cdot \nabla)\underset{\sim}{v}_n = -\frac{1}{\rho}\nabla p - \frac{1-\epsilon}{\epsilon} s\nabla T - \nabla\Omega - \frac{1-\epsilon}{2}\nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2 - (\underset{\sim}{v}_n - \underset{\sim}{v}_s)\frac{\Gamma}{\epsilon\rho} , \tag{5}$$

$$\frac{\partial \underset{\sim}{v}_s}{\partial t} + (\underset{\sim}{v}_s \cdot \nabla)\underset{\sim}{v}_s = -\frac{1}{\rho}\nabla p + s\nabla T - \nabla\Omega + \frac{\epsilon}{2}\nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2 , \tag{6}$$

where p is the pressure; T, the temperature; s, the entropy per unit mass carried by the fluid; Ω, the potential of the external force field; $\epsilon = \frac{\rho_n}{\rho}$; and

$$\Gamma = \frac{\partial \rho_n}{\partial t} + \nabla\cdot(\rho_n \underset{\sim}{v}_n) = -\left[\frac{\partial \rho_s}{\partial t} + \nabla\cdot(\rho_s \underset{\sim}{v}_s)\right] ,$$

the source density of the normal fluid.

We need only to remark that besides the ideas of the two-fluid theory, the approaches and arguments used in establishing the previous equation are those usually employed in ordinary mechanics of continuum; and as in elsewhere, the ultimate justification is the agreement with the physical reality. In that respect, the theory is quite successful. The phenomena (i), (iii), (iv) and (v) are satisfactorily explainable by the linearized version of the above set of equations, mainly due to the pres-sence of the thermal-mechanical terms $s\nabla T$ in Eqs. (5) and (6). In equations (5) and (6), the term, $-(\underset{\sim}{v}_n - \underset{\sim}{v}_s)\frac{\Gamma}{\epsilon\rho}$, may be interpreted as the average increment of $\underset{\sim}{v}_n$ in unit time due to the interactions between normal and superfluid "particles", while $\frac{1}{2}(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2$ is $\left(\frac{\partial U}{\partial \epsilon}\right)_{\rho, s}$, $U(\rho, s, \epsilon)$ being the internal energy density of the fluid[17].

Multiply (5) by ρ_n, (6) by ρ_s and then add, we obtain:

$$\frac{D\underset{\sim}{v}}{Dt} + \frac{1}{\rho}\nabla\cdot[\rho_n \underset{\sim}{v}_n \underset{\sim}{v}_n + \rho_s \underset{\sim}{v}_s \underset{\sim}{v}_s - \rho \underset{\sim}{v}\underset{\sim}{v}] = -\frac{1}{\rho}\nabla p - \nabla\Omega , \tag{7}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\underset{\sim}{v} \cdot \nabla)$ is the particle derivative with respect to time.

Multiply (5) by $\rho_n \underset{\sim}{v}_n$, (6) $\rho_s \underset{\sim}{v}_s$, and then add, we obtain

$$\frac{D}{Dt}\left[\frac{\epsilon v_n^2}{2} + \frac{(1-\epsilon)}{2} v_s^2 + U\right] = -\frac{1}{\rho} \nabla\cdot\left[\frac{\rho_n v_n^2}{2} (\underset{\sim}{v}_n - \underset{\sim}{v}) + \frac{\rho_s v_s^2}{2} (\underset{\sim}{v}_s - \underset{\sim}{v}) + p\underset{\sim}{v}\right.$$

$$\left. + \rho s T(\underset{\sim}{v}_n - \underset{\sim}{v}) + \rho\epsilon(1-\epsilon)\left(\frac{\partial U}{\partial \epsilon}\right)(\underset{\sim}{v}_n - \underset{\sim}{v}_s)\right] - \underset{\sim}{v}\cdot\nabla\Omega \quad , \tag{8}$$

where the pressure and temperature are related to U by

$$p = \rho^2 \left(\frac{\partial U}{\partial \rho}\right)_{s,\epsilon} \quad \text{and} \quad T = \left(\frac{\partial U}{\partial s}\right)_{\rho,\epsilon} \quad .$$

We may note that the divergence term in the left-hand side of (7) is the apparent stress due to diffusive transfer of momentum, which appears in every mixture. The divergence term in the right-hand side of (8), arises from the energy flux and the physical meanings of various terms are quite clear.

Equations (7) and (8) represents the principles of conservation of momentum and energy, and are obtainable independent of the mechanism of interaction between normal and superfluid "particles". On the other hand, the terms with Γ and $\nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2$ in Eqs. (5) and (6) arises because of the assumption of a particular kind of interaction, i. e., the superfluid velocity $\underset{\sim}{v}_s$ is not a thermal average, thus all superfluid "particles" should have the same velocity $\underset{\sim}{v}_s$ and any change of $\underset{\sim}{v}_s$ due to interactions would be a change for all "particles". The validity of this assumption has not been explicitly verified experimentally. The experimental study of second sound, being of small amplitude, serves only to establish the terms $s\nabla T$ in Eqs. (5) and (6). Here we suggest possible ways to test its validity, i. e. the study of propagation of finite amplitude waves.

For simplicity, let us make the approximations

$$p = p(\rho) \quad , \qquad T = T(s) \quad ,$$

and limit the considerations in such temperature range that it is approximately true $\frac{\epsilon}{s}$ is a constant. Then we can show that the characteristic speeds c satisfy the following equation: (Appendix A)

$$\begin{aligned} c^4 + c^3 [2(u+v)] + c^2 [u^2+v^2+4uv-a^2-b^2] + c\{2uv(u+v) - 2[(1-\epsilon)u+\epsilon v]a^2 \\ -2[\epsilon u+(1-\epsilon)v]\, b^2\} \\ + \{u^2v^2 - [(1-\epsilon)u^2+\epsilon v^2]\, a^2 - [\epsilon u^2+(1-\epsilon)v^2]\, b^2 + a^2b^2\} \\ +(u-v)\{c^3 + c^2[(1+\epsilon)u+(2-\epsilon)v] + c[\epsilon u^2+(1-\epsilon)v^2+2uv-a^2] + [\epsilon u^2 v+(1-\epsilon)uv^2 \\ -a^2(u-\epsilon u+\epsilon v)]\} = 0 \quad , \end{aligned} \tag{9}$$

where a and b are the speeds of first and second sound respectively, and u and v are the components of velocities of normal and super fluids in the direction normal to the characteristic surface. The last term with the factor $(u-v)$ would be missing if terms with $\nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2$ are missing in Eqs. (5) and (6). Thus the determination of the characteristic speeds can tell whether these terms are there.

We note that if u and v are small in comparison with both a and b, (9) will reduce to

$$(c^2 - a^2)(c^2 - b^2) = 0 \quad ,$$

leading to the acoustic limit, independent of the presence of the term with factor $(u-v)$ in Eq. (9). That is why we need the study of finite amplitude waves to decide the issue.

When u and v are small in comparison with a, but not necessarily b, i.e. when the fluid may be considered as incompressible, it may be obtained from (9) that

$$c = \frac{1}{2}\ \{-[(3-2\epsilon)u+(2\epsilon-1)v]\ \pm [(1-2\epsilon)^2(u-v)^2+4b^2]^{\frac{1}{2}}\} \quad . \tag{10}$$

On the other hand, if the last term with factor $(u-v)$ in (9) is missing, we obtain

$$c = -[(1-\epsilon)u + \epsilon v] \pm [b^2 - \epsilon(1-\epsilon)(u+v)^2]^{\frac{1}{2}} \quad . \tag{11}$$

Equation (10) implies that c will always be real, while Eq. (11) may permit complex values of c. As the complex value of c implies the instability of the system, just like the case of ordinary barotropic fluid if it should turn out that $\frac{dp}{d\rho} < 0$. Therefore it is more likely that the terms $\nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2$ should be present in Eqs. (5) and (6), even from a continuum mechanics point of view.

In addition to Eqs. (3) - (6), a further condition of local irrotationality is usually imposed on the superfluid component:

$$\nabla \times \underset{\sim}{v}_s = 0 \quad . \tag{12}$$

This condition like Eqs. (5) and (6) can also be established from two approaches: One from the variational principle of a macroscopic fluid(16),(17). The other based on the microscopic argument that since He^4 are Bose particles, therefore there are only very few excitations at low energy to keep the meaningful existence of the superfluid component. The "superfluid" component, being a giant quantum system with concerted motion, is by implication irrotational(18).

Lin(19) criticized both approaches. In particular, he pointed out that in the variational approach, an additional condition to insure the definite identification of the fluid particles should be imposed. Then no restriction on the irrotationality of superfluid component would follow. Though it is a well thought point, and gives correct formulation for ordinary compressible fluids, here it may also be argued that we should not apply this extra condition because He^4 are Bose particles. Again, we may need experiments for the final clarification on this issue.

III. Irreversible Hydrodynamic Equations.

The fundamental Eqs. (3) - (6) will be modified in order to account for the irreversible process. Equation (4) should be changed to

$$\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s \underset{\sim}{v}_n + \underset{\sim}{q}) = Q \quad , \tag{13}$$

where $\underset{\sim}{q}$ represents the non-convective heat flux, and Q the entropy production due to ordinary thermal conduction and frictional dissipations. To the right hand side of (5) and (6), we should add $\frac{1}{\rho_n}[-\underset{\sim}{F}_n + \underset{\sim}{F}_{sn}]$, and $\frac{1}{\rho_s}[-\underset{\sim}{F}_s - \underset{\sim}{F}_{sn}]$, respectively, where $\underset{\sim}{F}_n$ is the frictional force term for the normal component, $\underset{\sim}{F}_s$, the frictional force term for the superfluid component, and $\underset{\sim}{F}_{sn}$ is the mutual friction term. The nature of these newly introduced terms is still largely an unsettled question.

The necessity to introduce these additional terms is again dictated by the experimental observations. The implication in the two fluid theory that normal component is <u>normal</u> leads to the term $\underset{\sim}{F}_n$. And $\underset{\sim}{F}_n$, without evidence to the contrary, is naturally assumed to arise from the ordinary viscous stresses. Similarly, we have the ordinary thermal conduction term in $\underset{\sim}{q}$.

It seems also natural to assume that $\underset{\sim}{F}_s = 0$ on account of the phenomenon of superfluidity. But, as pointed out by London[17] and Lin[11], the absence of $\underset{\sim}{F}_s$ may result from the irrotationality of the superfluid component rather than the absence of the viscosity. In this sense, shear strains in the superfluid component could still exist and momentum may be transmitted without causing dissipation. An experiment to settle this point has been suggested by London[17], and as yet has not been performed. The phenomenon of superfluidity may even be due to the inefficiency of interaction between the superfluid component and the wall of confinements, even though the superfluid component is neither inviscid nor intrinsically irrotational, as suggested by Lin[19].

The apparent loss of superfluidity when the flow speed of He II exceeds certain critical values leads to the introduction of mutual friction. Microscopically, the existence of critical velocities implies that there should be other types of low energy excitation besides phonons and rotons. This led to the development of the concept of quantized vortex lines[18], whose basic features have now been supported by experimental observations[20], [21]. Except where there are only a few vortex lines or rings, the idea of quantized vortex lines is not too helpful from macroscopic point of view. All it says is that when the flow is supercritical, the superfluid component is practically rotational and frictional. It does serve the basis of a kinetic model to derive the macroscopic mutual friction. However, when many such vortex lines are present, it is indeed very difficult to calculate the interaction out of their entanglements. Thus other forms of mutual friction have also been suggested directly from a semi-empirical basis.

In the following we shall describe and discuss three different formulation:

(i) The Gorter-Mellink formulation[22]. For this formulation, we take

$$\underset{\sim}{F}_s = 0 \tag{14}$$

$$\underset{\sim}{q} = \frac{\kappa}{T} \nabla T \tag{15}$$

$$(-F_n)_i = \frac{\partial}{\partial x_k}\left\{\eta\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_{n\ell}}{\partial x_\ell}\right) + \zeta\delta_{ik}\frac{\partial v_{n\ell}}{\partial x_\ell}\right\}, \quad i,k = 1,2,3 \tag{16}$$

$$\underset{\sim}{F}_{sn} = \alpha\rho(1-\epsilon)\epsilon \left| \underset{\sim}{v}_s - \underset{\sim}{v}_n \right|^2 (\underset{\sim}{v}_s - \underset{\sim}{v}_n) \tag{17}$$

and

$$Q = \frac{1}{T}\left\{\frac{\kappa}{T}(\nabla T)^2 + \zeta(\nabla\cdot\underset{\sim}{v}_n)^2 + \frac{\eta}{2}\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_{n\ell}}{\partial x_\ell}\right)^2 + \alpha\rho\epsilon(1-\epsilon)(\underset{\sim}{v}_s - \underset{\sim}{v}_n)^4\right\}, \tag{18}$$

where η and ζ are coefficients of shear and bulk viscosity respectively, and α is the coefficient of mutual friction.

The boundary conditions are such that relative to the boundary surface, the perpendicular component of the total mass flux be zero, the perpendicular component of the heat flux be continuous, and the tangential component of normal fluid velocity be zero.

We may remark that for problems involving bulk fluid flows, $\underset{\sim}{q}$ and Q can usually be neglected because of the extreme efficiency of heat transfer by the internal convection.

(ii) The formulation by Lin[19]. For the case of incompressible fluids, we have in this formulation:

$$-\underset{\sim}{F}_n = \eta^{(nn)} \nabla^2 \underset{\sim}{v}_n \quad , \tag{19}$$

$$-\underset{\sim}{F}_s = \eta^{(sn)} \nabla^2 \underset{\sim}{v}_n \quad , \tag{20}$$

and

$$\underset{\sim}{F}_{sn} = \eta^{(ns)} \nabla^2 \underset{\sim}{v}_s + \rho\epsilon(1-\epsilon)(\nabla \times \underset{\sim}{v}_s) \times (\underset{\sim}{v}_n - \underset{\sim}{v}_s) \quad . \tag{21}$$

This formulation recognizes neither irrotationality nor inviscidness in the superfluid component. The second term in (21) is, strictly speaking, not a mutual friction term, since it does not conbribute to dissipation. It is absent when the flow is irrotational. We put it here for the convenience of comparison. Disregard this term for the moment, we see just like $-\underset{\sim}{F}_n + \underset{\sim}{F}_{sn} = \eta^{(nn)} \nabla^2 \underset{\sim}{v}_n + \eta^{(ns)} \nabla^2 \underset{\sim}{v}_s$, we also have $-\underset{\sim}{F}_s - \underset{\sim}{F}_{sn} = \eta^{(sn)} \nabla^2 \underset{\sim}{v}_n + \eta^{(ss)} \nabla^2 \underset{\sim}{v}_s$, except an assumption $\eta^{(ss)} + \eta^{(ns)} = 0$ is made to have only the viscous effect of the normal component present in the equation governing the total fluid. Because of (21), we need additional boundary condition to govern the velocity of the superfluid component. Instead of the non-slip condition which governs the normal component, it is suggested that the tangential component of the shear stress vector will be directly related to the slip velocity:

$$\tau_\alpha^{(s)} = F^{(s)}(w^{(s)2}) w_\alpha^{(s)} \tag{22}$$

where $\underset{\sim}{\tau}$ is the stress vector at the boundary, with unit normal $\underset{\sim}{n}$, i.e. $\tau_i^{(s)} = n_j \left[\eta^{(sn)} \left(\frac{\partial v_{ni}}{\partial x_j} + \frac{\partial v_{nj}}{\partial x_i} \right) - \eta^{(ns)} \left(\frac{\partial v_{si}}{\partial x_j} + \frac{\partial v_{sj}}{\partial x_i} \right) \right]$, $\underset{\sim}{w}^{(s)}$ is the velocity of the superfluid component relative to the boundary, and the subscript

α signifies the tangential direction. Superfluidity at low speeds of He II suggests that $F^{(s)}$ will vanish with $w^{(s)}$, thus as an approximation we could put

$$\tau_{\alpha}^{(s)} = \beta \mid w^{(s)}|^2 w_{\alpha}^{(s)} \quad . \tag{23}$$

(iii) The formulation due to Hall and Vinen[23], and Bekarevich and Khalatnikov[24]. For this formulation, we have

$$\underset{\sim}{q} = \frac{\kappa}{T} \nabla T \quad , \tag{24}$$

$$-\underset{\sim}{F}_s = -\underset{\sim}{\omega} \times [\nabla \times \lambda \underset{\sim}{\nu}] \quad , \tag{25}$$

$$(-F_n)_i = \frac{\partial}{\partial x_k}\left\{\eta\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_{n\ell}}{\partial x_\ell}\right) + \zeta_1 \delta_{ik} \frac{\partial}{\partial x_\ell} \rho_s (v_{s\ell} - v_{n\ell}) \right.$$
$$\left. + \zeta_2 \delta_{ik} \frac{\partial v_{n\ell}}{\partial x_\ell}\right\} \quad , \tag{26}$$

$$\underset{\sim}{F}_{sn} = -[\rho_s \nabla\{\zeta_3 \nabla\cdot\rho_s(\underset{\sim}{v}_s - \underset{\sim}{v}_n) + \zeta_1 \nabla\cdot\underset{\sim}{v}_n\}]$$
$$+ [B_1 \underset{\sim}{\omega} \times \underset{\sim}{\xi} + B_2 \underset{\sim}{\nu} \times (\underset{\sim}{\omega} \times \underset{\sim}{\xi}) - B_3 \underset{\sim}{\nu}(\underset{\sim}{\omega}\cdot\underset{\sim}{\xi})] \quad , \tag{27}$$

$$Q = \frac{1}{T}\left\{\kappa \frac{(\nabla T)^2}{T} + \frac{\eta}{2}\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_{n\ell}}{\partial x_\ell}\right)^2 + \zeta_2 (\nabla\cdot\underset{\sim}{v}_n)^2 \right.$$
$$+ \zeta_3 [\nabla\cdot\rho_s(\underset{\sim}{v}_s - \underset{\sim}{v}_n)]^2 + 2\zeta_1 (\nabla\cdot\underset{\sim}{v}_n)[\nabla\cdot\rho_s(\underset{\sim}{v}_s - \underset{\sim}{v}_n)]$$
$$\left. + \frac{B_2}{|\underset{\sim}{\omega}|}(\underset{\sim}{\omega} \times \underset{\sim}{\xi})^2 + \frac{B_3}{|\underset{\sim}{\omega}|}(\underset{\sim}{\omega}\cdot\underset{\sim}{\xi})^2\right\} \quad , \tag{28}$$

where $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{v}_s$, $\underset{\sim}{\xi} = \underset{\sim}{v}_n - \underset{\sim}{v}_s - \frac{1}{\rho_s} \nabla \times \lambda \underset{\sim}{\nu}$, $\underset{\sim}{\nu} = \frac{\underset{\sim}{\omega}}{|\underset{\sim}{\omega}|}$, $\kappa, \eta, \zeta_2, \zeta_1, \zeta_3$, B_1, B_2, B_3 and λ are coefficients responsible for thermal conduction, viscosities and mutual frictions. For boundary condition, let N be the unit normal vector at the surface of the boundary, and u the velocity of the boundary, we then have

$$\left\{ \underset{\sim}{v}_s - \underset{\sim}{u} + \frac{1}{\rho_s} \nabla \times \lambda \underset{\sim}{\nu} + \frac{B_1}{\rho_s} \underset{\sim}{\xi} + \frac{B_2}{\rho_s} \underset{\sim}{\nu} \times \underset{\sim}{\xi} \right\} \times \underset{\sim}{\omega} = \zeta_b \underset{\sim}{N} \times \underset{\sim}{\omega} + \zeta_b' (\underset{\sim}{N} \times \underset{\sim}{\nu}) \times \underset{\sim}{\omega} \quad , \tag{29}$$

in addition to the non-slip condition for $\underset{\sim}{v}_n$, the vanishing and continuity of the perpendicular components of the total mass and heat fluxes. Equation (29) is derived from the consideration of the dissipation at the surface due to vortex slippage, and ζ_b and ζ_b' are the boundary dissipation coefficients.

This last formulation is certainly the most complete among the three, and it is also appealing, since essentially the same equations can be derived from either a continuum approach or a microscopic physical model. But it is also a set of very complex equations highly non-linear, and involving many undetermined physical coefficients, thus it is difficult to compare with specific results. In (27), the terms in the second bracket is present only when $\omega = 0$, this is the outcome of the rotation of the superfluid component or the quantized vortex lines. In the microscopic physical model, the force is transmitted from the normal component to the vortex lines through the collisions between rotons and the vortex lines, and then treating the vortex lines as some foreign filament with circulation, the force is transmitted to the superfluid component through the Magnus effect. The term associated with B_3 is longitudinal, i.e., the force is in the direction of vorticity, while the terms associated with B_1 and B_2 are transverse. Also the term associated with B_1 does not contribute to the dissipation.

The mutual firction of Gorter-Mellink could be easily incorporated in the formulation HVBK, from a continuum point of view. But as they now stand, there is a fundamental difference. In the formulation HVBK, though it may not be possible to establish a theorem on the permanence of vorticity for the superfluid component, it is readily seen that irrotational flow is a permissible state of motion, while in the Gorter-Mellink formulation, the superfluid component cannot be rotation free because of the mutual friction.

Lin's formulation is quite different from the other two formulations, but from a continuum point of view, it has as legitimate a theoretical basis as the other two. This only shows how primitive is our knowledge about the hydrodynamics of He II. The nonlinear and irreversible aspects of the flow of He II are still very much unexplored.

A simple problem that may serve to show the different conclusions drawn from these three formulations is the steady parallel flow of He II through a circular pipe. (Appendix B).

For this problem, we assume that the fluids are incompressible, all the physical coefficients constant, and we look for flows such that in cylindrical coordinates (r, θ, z), we have

$$\underset{\sim}{v}_n = (0, 0, u(r)) \quad , \qquad \text{and} \quad \underset{\sim}{v}_s = (0, 0, v(r)) \quad .$$

Then it is found that for all three formulations,

$$\frac{\partial p}{\partial z} = -A \quad ,$$

which is a constant, and

$$u = \frac{A}{4\eta} (a^2 - r^2) \quad , \tag{30}$$

where a is the radius of the pipe, and in Lin's formulation we have denoted $\eta = \eta^{(nn)} + \eta^{(sn)}$. But the solutions for v_s are different. For Gorter-Mellink mutual friction, we have

$$v = \frac{A}{4\eta}(a^2 - r^2) + \left(\frac{A}{\alpha\rho\epsilon}\right)^{\frac{1}{3}} . \tag{31}$$

From Lin's formulation, we have

$$v = \frac{A}{4\eta^{(ns)}}\left[\frac{\eta^{(sn)}}{\eta} - (1-\epsilon)\right](a^2 - r^2) - \left[\frac{(1-\epsilon)aA}{2\beta}\right]^{\frac{1}{3}} ; \tag{32}$$

while, in HVBK formulation, v will satisfy the following equation:

$$\left[v + \frac{A}{4\eta}(r^2 - a^2) + \frac{\lambda}{\rho_s}\frac{1}{r}\right]\frac{dv}{dr} + \frac{A(1-\epsilon)}{B_2} = 0 . \tag{33}$$

Moreover, the temperature T comes out to be constant automatically in the Gorter-Mellink formulation, while the restriction of constant temperature will lead to inconsistencies in the other two formulations, and there we have $T = T(r)$. In the HVBK formulation, the pressure p will also vary with r, while in the other two formulations, $p = p(z)$.

The solutions both (31) and (32) represent parabolic velocity profiles. The solution of (33), though not readily obtainable, is definitely not parabolic. In particular,

$$\left(\frac{d^3 v}{dr^3}\right)_{r=o} = \frac{2A(1-\epsilon)^3 \rho^2}{B_2 \lambda^2}\left[v(o) - \frac{Aa^2}{4\eta}\right] . \tag{34}$$

The measurement of the curvature of this velocity profile in the central portion of the pipe is a possible way to test the validity of these formulations.

IV. Flow with Gorter-Mellink Mutual Friction.

The Gorter-Mellink formulation, though lacking direct microscopic physical basis, has the virtue of relative simplicity. Its semi-empirical nature is also an indication of its reliability. It should therefore serve as a good starting point for the theoretical analysis of the hydrodynamic problems in superfluid helium.

Let us first consider the case in which the flow speeds are low enough so that the fluids can be treated as incompressible and the entropy density s can also be taken as constant. We also like to neglect the thermal conduction and dissipation term since the internal convection will be the dominating mechanism for heat transfer. Then, the fundamental equations become

$$\nabla \cdot \underset{\sim}{v}_n = 0 \quad , \tag{35}$$

$$\nabla \cdot \underset{\sim}{v}_s = 0 \quad , \tag{36}$$

$$\frac{\partial \underset{\sim}{v}_n}{\partial t} + (\underset{\sim}{v}_n \nabla) \underset{\sim}{v}_n = - \frac{1}{\rho} \nabla p - \frac{1-\epsilon}{\epsilon} s \nabla T - \nabla \Omega - \frac{1-\epsilon}{2} \nabla (\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2 + \frac{\eta}{\epsilon\rho} \nabla^2 \underset{\sim}{v}_n$$

$$+ \alpha(1-\epsilon)\,(\underset{\sim}{v}_s - \underset{\sim}{v}_n)^3 \quad , \tag{37}$$

and

$$\frac{\partial \underset{\sim}{v}_s}{\partial t} + (\underset{\sim}{v}_s \cdot \nabla) \underset{\sim}{v}_s = - \frac{1}{\rho} \nabla p + s \nabla T - \nabla \Omega + \frac{\epsilon}{2} \nabla (\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2 - \alpha\epsilon\,(\underset{\sim}{v}_s - \underset{\sim}{v}_n)^3 \quad . \tag{38}$$

In the following, we would like to report the results of some analyses based on the above set of equations.

(i) Some exact solutions[25].

For steady flows, the following exact solutions are readily obtained:

(a) Flow through a pipe of elliptical section.

Let the section be represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1 \quad .$$

Denote $A = -\frac{d}{dz}(p + \rho\Omega)$, $B = \rho s \frac{dT}{dz}$. A and B are both constants, and z is in the direction of flow. Then with

$$\underset{\sim}{v}_n = (0, 0, u) \qquad \text{and} \qquad \underset{\sim}{v}_s = (0, 0, v) \quad ,$$

we have

$$u = \frac{Aa^2 b^2}{2\eta\,(a^2 + b^2)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) , \tag{39}$$

$$v = u + \left(\frac{A+B}{\alpha\epsilon\rho}\right)^{\frac{1}{3}} \quad , \tag{40}$$

and the total discharge rate is

$$Q = \rho\pi ab\left[\frac{Aa^2b^2}{4(a^2 + b^2)\eta} + (1 - \epsilon)\left(\frac{A+B}{\alpha\epsilon\,\rho}\right)^{\frac{1}{3}}\right] . \tag{41}$$

(b) Flow through a pipe of annular section.

Let the section be represented by $b \leqslant r \leqslant a$. Then with the same general notation as in (a), we obtain

$$u = \frac{A}{4\eta}\left(b^2 - r^2 + \frac{a^2 - b^2}{\log a/b}\,\log\frac{r}{b}\right) , \tag{42}$$

$$v = u + \left(\frac{A+B}{\alpha\epsilon\,\rho}\right)^{\frac{1}{3}} \quad , \tag{43}$$

and

$$Q = \pi(a^2 - b^2)\left[\frac{A}{8\eta}\left(a^2 + b^2 - \frac{a^2 - b^2}{\log a/b}\right) + (1 - \epsilon)\left(\frac{A+B}{\alpha\epsilon\rho}\right)^{\frac{1}{3}}\right] . \tag{44}$$

(c) Plane Couette Flow.

Let the channel be $-d < y < d$, and the plane $y = d$ moving with u_o in the z-direction. Then with the same general notation as in (a), we obtain

$$u = \frac{A}{2\eta}(d^2-y^2) + \frac{u_o}{2}\left(1+\frac{y}{d}\right), \tag{45}$$

$$v = u + \left(\frac{A+B}{\alpha\epsilon\rho}\right)^{\frac{1}{3}}. \tag{46}$$

(d) Cylindrical Couette Flow.

Let the radii of the rotating inner and outer cylinder be r_1 and r_2 and rotate with angular velocities ω_1 and ω_2 respectively. Then, in cylindrical coordinates (r, θ, z), with

$$\underset{\sim}{v}_n = (0, u(r), 0) \quad , \qquad \text{and} \qquad \underset{\sim}{v}_s = (o, v(r), 0) \quad ,$$

we obtain, after neglecting Ω:

$$T = \text{const.} ,$$

$$\frac{dp}{dr} = \frac{u^2}{r} , \tag{47}$$

and

$$u = v = \frac{1}{r_2^2 - r_1^2}\left[r(\omega_2 r_2^2 - \omega_1 r_1^2) - \frac{r_1^2 r_2^2}{r}(\omega_2 - \omega_1)\right]. \tag{48}$$

Exact solutions for flow with suction and flow in convergent and divergent channels can also be obtained.

(ii) The boundary layer[25].

For two dimensional steady boundary layer over a flat surface, let us designate the coordinate along the surface be x, and the perpendicular to the surface y. The outer flows are assumed to be

$$\underset{\sim}{v}_n = \underset{\sim}{v}_s = (U(x),\ 0) \quad ,$$

and in the boundary layer, let

$$\underset{\sim}{v}_n = (u_x, u_y) \quad \text{and} \quad \underset{\sim}{v}_s = (v_x, v_y) \quad .$$

Then we have, outside the boundary layer

$$T = T_o \quad ,$$

and

$$p + \frac{\rho}{2} U^2 = \text{const.} \tag{50}$$

Inside the boundary layer, we have

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \quad , \tag{51}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad , \tag{52}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = U \frac{dU}{dx} + \frac{\eta}{\epsilon \rho} \frac{\partial^2 u_x}{\partial y^2} + \alpha(1-\epsilon)\,(v_x - u_x)^3 \quad , \tag{53}$$

and

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = U \frac{dU}{dx} - \alpha\epsilon(v_x - u_x)^3 \quad . \tag{54}$$

The pressure inside the boundary layer will again be given by (50), while the temperature variation by

$$T = T_o - \frac{\epsilon}{2s} (u_x - v_x)^2 \quad . \tag{55}$$

If we look for similarity solutions of the form:

$$u_x = Uf'(\eta) \quad , \quad v_x = Ug'(\eta)$$

with

$$\eta = \frac{y}{h(x)} \quad ,$$

it turns out that the only permissible solutions are those with

$$U = -\frac{a}{x} \quad , \quad \text{and} \quad h = bx \quad , \tag{56}$$

which correspond to the flows in the convergent and divergent channels. Thus although there does exist similarity solution, the varities are much more limited than those of ordinary viscous fluids.

The boundary layer like that of ordinary fluids, tends to separate when there is an adverse pressure gradient along the wall. The point of separation for the normal component, P, is again determined by the condition

$$\left(\frac{\partial u_x}{\partial y}\right)_P = 0 \quad , \tag{57}$$

and backflow appears downstream from P. But the appearance of this separation point is somewhat delayed, compared with corresponding situations for ordinary fluid flow because of the forward drag of superfluid component by the action of the mutual friction. The superfluid component will slip over the surface of the wall with speed reduced from U due to the mutual friction. A minimum in velocity profile will appear beyond P, and eventually at a point Q, some distance from the downstream, the streamline will divide, and backflow of the superfluid component will also appear beyond this point. The qualitative picture of the flow configuration is shown in Fig. 1. Inasmuch as the boundary layer will not remain thin once any separation occurs, the controlling point of separation should be that of the normal component.

(iii) Stability of Flow Down an Inclined Plane[26].

Let the angle that an inclined plane makes with the horizontal be θ; then the velocity fields of a fluid layer flowing under gravity g down the plane are given by:

$$\underset{\sim}{v}_n = (U, 0, 0) \quad , \quad v_s = (V, 0, 0)$$

with

$$U = (\rho g \sin\theta / 2\eta)\,(h^2 - y^2) \quad ,$$

and

$$V = U + \left(\frac{g \sin\theta}{\alpha\epsilon}\right)^{\frac{1}{3}} \quad ,$$

where $y = 0$ defines the free surface, and h is the depth of the fluid layer.

It may be shown that the stability of the flow is governed by the long wave length disturbances. Denote $R_o = U(o)h\rho/\eta$, $R' = \frac{\epsilon}{2}\left(\frac{U-V}{U(o)}\right)R$, then it is found that the flow will be unstable if

$$\frac{8}{5}\,\epsilon R_o + \frac{4}{3}\,(1-\epsilon)R' - 2\epsilon\cot\theta - 2(1-\epsilon)R'\left(\frac{R'}{R_o}\right) - 2(1-\epsilon)\left(\frac{R'}{R_o}\right)\cot\theta > 0 \ . \tag{58}$$

In the limit of $\epsilon \to 1$, (58) becomes

$$R_o > \frac{5}{4}\cot\theta \quad ; \tag{59}$$

while as $\epsilon \to 0$, (58) becomes

$$R_o > \frac{3}{2} \cot\theta \quad . \tag{60}$$

The essential mechanism of this type of instability is inherent in the inviscid flow[27]. Viscosity only plays the role to produce the velocity profile of the primary flow. Then it is not surprising that for both limits $\epsilon \to 1$ and $\epsilon \to 0$, the Reynolds number based on the total fluid density is the relevant parameter. Another point of interest which can be deduced from (58) is that there exists flow configurations such that the system is stable even for $\theta > \frac{\pi}{2}$, i.e. when the fluid is flowing under the plane. For He II at $T = 1.4°K$, the situation is realizable for $h \lesssim 10^{-4}$ cm. This feature should have some bearing on the nature of the film flow. Or, the study of film flow in this respect should shed some light on the understanding of the nature of the mutual friction.

(iv) Rectified Internal Convection and Ultrasonic Cavitation[28].

The onset of cavitation bubbles in ordinary liquids is closely related to the phenomenom of rectified diffusion of mass into the bubble. For He II, by far the dominating mechanism of heat transfer is the internal convection, and since it is a convection, mass is transferred with the heat. Then it is found that for oscillating bubbles in the liquid helium, we have the analogous phenomenon of rectified internal convection. The threshold pressure for the cavitation can then be calculated, and it is found, that for $T > 1.6K$, yet away from T_λ, the threshold pressure may be approximately given as

$$p_t = \left[\frac{12(1-\epsilon)p\eta}{R_o}\right]^{\frac{1}{2}} \left[\frac{10\sigma s}{\epsilon \alpha R_o^{\,2}}\left(\frac{dT}{dp}\right)\right]^{1/6} \quad , \tag{61}$$

where p is the vapor pressure, σ, the surface tension, and R_o, the radius of the bubble nucleus. The results agrees fairly well with the observation,[29] if R_o is taken to be 3×10^{-4} cm.

The more important problem about the nature of the cavitation nuclei is unanswered in this investigation. Its understanding should improve a great deal our knowledge of the superfluid as well as ordinary liquids.

(v). Nonlinear internal convection.

The previous analyses are based on the assumption that the temperature variation in the region of interest is not large, hence ϵ, s, and all the physical coefficients can be taken as constant. Since these parameters usually vary quite sensitively with temperature, we have to take their variation into account when the temperature variation in the flow region is not small. Let us consider then the simplest problem that incorporates these effects, i.e. the problem of steady one-dimensional internal convection.

Take

$$\underset{\sim}{v}_n = (u(x),\ 0, 0) \quad , \quad \text{and} \quad \underset{\sim}{v}_s = (v(x), 0, 0) \quad ,$$

and also take ρ as constant, and neglect the thermal conduction and dissipation. We then obtain

$$su = a \quad , \tag{62}$$

and

$$\epsilon u + (1-\epsilon)v = \text{constant},$$

which we shall take as zero to concentrate on the aspect of internal convection, and a is a constant. Denote $\nu = \dfrac{2\eta + \zeta}{\rho}$, then we obtain

$$\frac{d^2u}{dx^2} = G(u) + F(u)\,\frac{du}{dx} \quad , \tag{63}$$

where

$$F(u) = \frac{\epsilon}{\nu}\left[\frac{3u}{1-\epsilon} + \frac{u^2}{(1-\epsilon)^2\epsilon}\,\frac{d\epsilon}{du} + \frac{s}{\epsilon}\,\frac{dT}{du}\right] \quad ,$$

and

$$G(u) = \frac{\alpha\epsilon u^3}{\nu(1-\epsilon\;)^3} \quad ,$$

since $\quad T = T(s) = T(u;a) \quad , \qquad$ and $\qquad \epsilon = \epsilon\,(T) = \epsilon(u;a) \quad .$

Equation (63)can be numerically integrated without difficulty, when appropriate boundary conditions are applied.

Here it is illuminating to compare this problem with the analogous problem of a barotropic fluid. There we obtain

$$\rho u = \text{constant},$$

and

$$\frac{d^2u}{dx^2} = \frac{1}{\nu}\left[u + \frac{1}{\rho}\,\frac{dp}{d\rho}\,\frac{d\rho}{du}\right]\frac{du}{dx} \quad . \tag{64}$$

From (64), it will automatically follow, for cases that $p \sim \rho^{\gamma}$ with $\gamma > 1$, that the downstream flow behavior depends very much on whether the flow at the initial section is subsonic or supersonic. For subsonic initial flows, the flow downstream will also be subsonic and differs little in speed with the initial speed, implying that uniform flow would not be a bad approximate solution. For supersonic initial flows, the downstream flow can either become faster and faster indefinitely, or go through a narrow region of compression to a uniform subsonic flow. The former case corresponds to an expansion flow, and the latter, a shock. It is worth noting that it comes out automatically that the shock is always compressive, and the supersonic flow is always in the upstream of the stationary shock while the subsonic flow is in the downstream.

Equation (63) is more complex than (64), still some qualitative conclusion can be drawn from it. Somewhat analogous situation exists with the replacement of sound speed by some speed L, of the order of the speed of second sound. However, even for sub-L initial flows, downstream flow will not remain bounded near the initial speed indefinitely. This unstable feature is solely due to the presence of mutual friction. For super-L initial flows, we also have either a downstream flow, transferring heat faster and faster, or the appearance of shock, whose downstream, however, will not stay sub-L indefinitely.

Khalatnikov[30], in his analysis of the progressive distortion of finite amplitude waves into shocks, found that for barotropic fluids, the shock will be compressive or expansive when $c + \rho \frac{dc}{d\rho}$ is positive or not, while for He II, the temperature shock will be heating or cooling when $\frac{d}{dT}\left[\frac{1-\epsilon}{\epsilon} s^2 c_2\right]$ is positive or not, where c and c_2 are speeds of sound and second sound respectively. These results are also contained in the solutions of (63) and (64). Indeed, the reason that only compressive shocks exist for ordinary fluid is due to the fact that we always have $c + \rho \frac{dc}{d\rho} > 0$. The same can not be said of $\frac{d}{dT}\left[\frac{1-\epsilon}{\epsilon} s^2 c_2\right]$. In fact, for temperature above 2.0°K and in the interval between 0.4 to 0.9°K, we have $\frac{d}{dT}\left[\frac{1-\epsilon}{\epsilon} s^2 c_2\right] < 0$. Then we should have a sub-L upstream flow going through a stationary cooling shock to a super-L downstream flow. The detailed experimental study of nonlinear heat transfer in different temperature range should be very helpful to our understanding of the hydrodynamics of superfluid helium.

V. Discussions and Speculations.

The uncertainty about the hydrodynamic formulation of superfluid helium is in part a reflection of the incompleteness of our understanding of the microscopic physics of the liquid helium. The prevailing theory initiated by Landau(12) can give satisfactory explanation of phenomena only up to temperature not too close to λ-point. It fails completely to account for the singular behavior in the λ-transition. Now, in this theory, excitations corresponding to different parts of a single spectrum are identified with phonons and rotons. Phonons represent the sound waves, while the rotons, as Feynman and Cohen(31) pointed out, will generate flow field very much like that due to a tiny classical vortex ring. This leads to the obvioud question: are rotons vortex rings? If so, presumably, they are quantized vortex rings(21). Then the other type of excitation, i.e., the quantized vortex lines, may just be another aspect of the rotons. Some very interesting features come out if we pursue along this direction somewhat further.

We may take the view that there are two distinct types of excitations, i.e. phonons, and rotons, each having its own spectrum. For phonons, we have the dispersion relation:

$$\epsilon^{(p)} = cp \quad , \tag{65}$$

where p is the momentum, and c, the sound speed. While for rotons, as suggested by the classical hydrodynamics, the dispersion relation can be taken as(21):

$$\epsilon^{(r)} = Ap^{\frac{1}{2}} \tag{66}$$

where

$$A = \frac{\delta}{2}\left(\frac{\rho K^3}{\pi}\right)^{\frac{1}{2}} \quad , \tag{67}$$

K is the circulation, and δ is a constant of order unity, which will depend on the size of the vortex ring and the nature of its core. Assume the circulations are quantized, then for He^4 with atomic mass m, we have

$$K = \frac{h}{m} = 0.997 \times 10^{-3} cm^2 sec^{-1} \quad .$$

Most vortex rings would have only one unit of circulation, since for the same momentum, to have two units of circulation would increase the energy by eight-fold. Formally, for an assembly of multitudes of phonons and rotons, the energy of a given state may be schematically written as

$$E = E^{(p)} + E^{(r)} + E^{(pr)} \quad , \tag{68}$$

where $E^{(p)}$ is the energy due to phonons if no rotons are present, $E^{(r)}$ that due to rotons if no phonons are present, and $E^{(pr)}$, the remaining part which may be called the phonon-roton interaction energy. Let us neglect $E^{(pr)}$ as a first approximation. In the same approximation, we shall neglect the interactions among phonons, then

$$E^{(p)} = \sum_i n_i^{(p)} \epsilon_i^{(p)} = \sum_i n_i^{(p)} c\, p_i \quad , \tag{69}$$

where $n_i^{(p)}$ is the number of phonons with momentum p_i.

The expression of $E^{(r)}$ will not be as simple as that of $E^{(p)}$. Again, from classical hydrodynamics, it is shown[32] that the energy of a system of circular vortex rings is

$$T = \sum_i \left[2p_i v_i' - \underset{\sim}{R}_i \cdot \frac{d\underset{\sim}{p}_i}{dt} \right] + \frac{\rho}{2} \iint_s V^2 \underset{\sim}{r} \cdot \underset{\sim}{n} \, dS \quad , \tag{70}$$

where $\underset{\sim}{V}$ is the velocity of the fluid, and $\underset{\sim}{R}_i$ is position vector of the center of the i^{th} vortex ring, v_i' is the average velocity of the i^{th} vortex ring in the direction normal to the plane of the vortex ring, and p_i is the momentum of the i^{th} vortex ring as if it is single.

The last term of (70) will yield a term like $\frac{1}{2} M\overline{V}^2$, where M is the total mass of the fluid and $\overline{V^2}$ is the average of V^2 over the boundary. It is essentially a constant, hence may be dropped from the expression. The term $\sum_i \underset{\sim}{R}_i \cdot \frac{d\underset{\sim}{p}_i}{dt}$ may be interpreted as that due to collisional interactions, which we shall neglect in consistence with the neglect of $E^{(pr)}$ and the phonon-phonon interaction energies. Then we have

$$E^{(r)} = \sum_j 2p_j(v_j + w_j) \quad ,$$

where v_j is the velocity of the j^{th} roton as if it is single, and w_j the average velocity in the direction of v_j induced by all the rest of the vortex rings. Or we may rewrite the last equation as

$$E^{(r)} = \sum_i n_i^{(r)} \left[Ap_i^{\frac{1}{2}} + 2p_i u_i \right] \quad , \tag{71}$$

where $n_i^{(r)}$ is the number of rotons with momentum p_i, and u_i is the average of w's over these $n_i^{(r)}$ rotons.

From (71), we see that the energy of a state will not only depend on the distributions in numbers of rotons, $\{n_i^{(r)}\}$, but also on the arrangements and orientations of $\{P\}$ of the vortex rings. We may thus write

$$E\{n_i^{(p)}, n_i^{(r)}, P\} = E_o + \sum_{p_i} n_i^{(p)} cp_i + \sum_{p_j} n_j^{(r)} A p_j^{\frac{1}{2}} + \sum_{p_j} 2n_j^{(r)} p_j u_j(P) \ . \qquad (72)$$

The partition function Q is thus

$$Q = \sum_{\{n_i^{(p)}, n_j^{(r)}, P\}} \exp[-E\{n_i^{(p)}, n_j^{(r)}, P\}/kT] \ . \qquad (73)$$

Let us denote

$$q = \sum_{\{P\}} \exp\left[-2\sum_{p_j} n_j^{(r)} p_j u_j(P)/kT\right] . \qquad (74)$$

In general, q will depend on $\{n_j^{(r)}\}$. But it is conceivable that q may not depend on $\{n_j^{(r)}\}$ sensitively. Rather it may only depend on the total number of rotons present which is related directly to the density and temperature of the system. If that is the case, then q may be factored out, and (73) becomes

$$Q = qe^{-E_o/kT} \cdot \prod_{p_i} \frac{1}{1-e^{-cp_i/kT}} \cdot \prod_{p_j} \frac{1}{1-e^{-Ap_j^{\frac{1}{2}}/kT}} , \qquad (75)$$

where the range of p_i and p_j in the products can be determined by arguments like those in the Debye's theory of solids.

The information about the λ-transition is now contained in the expression of q. First to note is its resemblance to the partition function of the Ising problem. For two-dimensional Ising problem with

$$E\{s_i\} = -\sum_{<ij>} \epsilon s_i s_j \quad ,$$

where s_i can take values either $+1$ or -1, $<ij>$ denotes a nearest-neighbor pair of spins, and $\epsilon > 0$, it is found[33] that the specific heat in the neighborhood of the transition temperature T_c is

$$C = -kC_c \ell n \mid T - T_c \mid \quad ,$$

with

$$\begin{aligned} C_c &= 0.4781 && \text{for hexagonal lattice,} \\ &= 0.4945 && \text{for square lattice,} \\ &= 0.4991 && \text{for triangular lattice.} \end{aligned}$$

For liquid helium, the singular part of the specific heat per atom near T_λ is given by[5]

$$C = -0.63k \ell n \mid T - T_\lambda \mid \quad .$$

The calculation of q of course is much more complex than the two-dimensional Ising problem. It is a three dimensional problem with not merely nearest neighbor interaction. Moreover, the vortex rings are not fixed in space and their orientations are not necessarily quantized. However, for their lowest energy configuration, the pattern may be simply orientated and relatively stationary thus not too different from that of Ising problem.

Within this model, the phenomenon of superfluidity may be interpreted as follows. Below λ-point, there is a long range order among roton's arrangements and orientations. The disturbances from any external agents tend more favorably to create new rotons rather than change the energies and momenta of existing rotons and destroy the long range order. Thus we have the superfluidity for flows below the critical velocity needed to create the most easily exictable excitations, i.e., the quantized vortex lines, or in our terminology, the large rotons. Above the λ-point, no long range order exists for rotons. Thus although new excitations could be created by external disturbances, their momenta and energies may be preferably spent to change the momenta and energies of the existing rotons. Then superfluidity will disappear.

The difficulty in the calculation of q prevent us to make any quantitative test of this model. This qualitative speculation is just the first step towards a fuller understanding of the problem. In larger aspects, it has long been our conviction that a successful theory of liquid structure may be hinged on the discovery of the right elementary excitations just like what phonons are to solids. Thus, it may be worthwhile to explore this idea of roton theory even for ordinary liquids.

Appendix A

In addition to Eqs. (3) - (6), we also need equations of state, which, for simplicity we assume to be

$$p = p(\rho) \qquad \text{and} \qquad T = T(s) \quad .$$

Denote $\dfrac{dp}{d\rho} = a^2$ and $\dfrac{(1-\epsilon)s^2}{\epsilon}\left(\dfrac{dT}{ds}\right) = b^2$,

where a is the speed of first sound and b, the speed of second sound.

It is approximately true in the temperature range $T = 1.4°K$ - $2.19°K$ that

$$\epsilon = \beta s \quad ,$$

where β is a constant[17]. We shall make this simplification. Then Eq. (4) is equivalent to $\Gamma = 0$.

Rewrite Eqs. (3) and (4) as:

$$\frac{\partial\rho}{\partial t} + \beta s\rho\nabla\cdot(\underset{\sim}{v}_n-\underset{\sim}{v}_s)+\beta s(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\cdot\nabla\rho+\beta\rho(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\cdot\nabla s+\rho\nabla\cdot\underset{\sim}{v}_s+(\underset{\sim}{v}_s\cdot\nabla)\rho = 0 \;, \tag{A1}$$

and

$$s\frac{\partial\rho}{\partial t} + \rho\frac{\partial s}{\partial t} + \rho s(\nabla\cdot\underset{\sim}{v}_n) + \rho(\underset{\sim}{v}_n\cdot\nabla)s + s(\underset{\sim}{v}_n\cdot\nabla)\rho = 0 \quad . \tag{A2}$$

Let $\varphi(\underset{\sim}{x}, t) = \text{const.}$ be a family of surface, and let $\delta\rho, \delta s, \delta\underset{\sim}{v}_n$, $\delta\underset{\sim}{v}_s$ be the jump in $\rho, s, \underset{\sim}{v}_n$ and $\underset{\sim}{v}_s$ across these surfaces in the direction of increasing value of ϕ. Then from (A1), (A2), (5) and (6) we obtain, after neglecting Ω:

$$\{\varphi_t+[\beta s(\underset{\sim}{v}_n-\underset{\sim}{v}_s)+\underset{\sim}{v}_s]\cdot\nabla\varphi\}\,\delta\rho+\beta s\rho\nabla\varphi\cdot\delta\underset{\sim}{v}_n+\rho(1-\beta s)\nabla\varphi\cdot\delta\underset{\sim}{v}_s+\beta\rho(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\cdot\nabla\varphi\,\delta s = 0, \tag{A3}$$

$$s(\varphi_t+\underset{\sim}{v}_n\cdot\nabla\varphi)\delta\rho + \rho s\nabla\varphi\cdot\delta\underset{\sim}{v}_n + \rho(\varphi_t+\underset{\sim}{v}_n\cdot\nabla\varphi)\delta s = 0 \quad , \tag{A4}$$

$$\frac{a^2}{\rho}\nabla\varphi\,\delta\rho + [\varphi_t + \underset{\sim}{v}_n\cdot\nabla\varphi + (1-\epsilon)(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\cdot\nabla\varphi]\,\delta\underset{\sim}{v}_n - (1-\epsilon)(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\cdot\nabla\varphi\,\delta\underset{\sim}{v}_s$$

$$+ (1-\epsilon)(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\times(\nabla\varphi\times\delta\underset{\sim}{v}_n) - (1-\epsilon)(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\times(\nabla\varphi\times\delta\underset{\sim}{v}_s) + \frac{b^2}{s}\nabla\varphi\,\delta s = 0 \; , \tag{A5}$$

and

$$\frac{a^2}{\rho}\nabla\varphi\,\delta\rho - \epsilon(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\cdot\nabla\varphi\delta\underset{\sim}{v}_n + [\varphi_t+\underset{\sim}{v}_s\cdot\nabla\varphi + \epsilon(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\cdot\nabla\varphi]\,\delta\underset{\sim}{v}_s$$

$$-\epsilon(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\times(\nabla\varphi\times\delta\underset{\sim}{v}_n) + \epsilon(\underset{\sim}{v}_n-\underset{\sim}{v}_s)\times(\nabla\varphi\times\delta\underset{\sim}{v}_s) - \frac{\epsilon}{1-\epsilon}\,\frac{b^2}{s}\nabla\varphi\,\delta s = 0 \; . \tag{A6}$$

Then $\varphi(\underset{\sim}{x}, t)$ = const. will be characteristic surfaces, if non-trivial solutions for $\delta\rho, \delta s, \delta\underset{\sim}{v}_n$ and $\delta\underset{\sim}{v}_s$ of (A3) - (A6) exist. Let us denote $\dfrac{\varphi_t}{|\nabla\varphi|} = c$, and in terms of components in the orthogonal characteristic coordinates such that $\nabla\varphi = (|\nabla\varphi|, 0, 0)$, write $\underset{\sim}{v}_n = (u, u_1, u_2)$ and $\underset{\sim}{v}_s = (v, v_1, v_2)$. Then Eqs. (A3) - (A6) become:

$$\{c+[\epsilon u+(1-\epsilon)v]\}\,\delta\rho+\epsilon\rho\delta u+(1-\epsilon)\rho\delta v+\beta\rho(u-v)\delta s = 0$$

$$s(c+u)\delta\rho+s\rho\delta u+\rho(c+u)\delta s = 0$$

$$\frac{a^2}{\rho}\delta\rho+[c+u+(1-\epsilon)(u-v)]\,\delta u+(1-\epsilon)(u_1-v_1)\delta u_1+(1-\epsilon)(u_2-v_2)\delta u_2$$

$$-(1-\epsilon)(u-v)\delta v-(1-\epsilon)(u_1-v_1)\delta v_1-(1-\epsilon)(u_2-v_2)\delta v_2 + \frac{b^2}{s}\delta s = 0$$

$$(c+u)\delta u_1 = 0$$

$$(c+u)\delta u_2 = 0$$

$$\frac{a^2}{\rho}\,\delta\rho-\epsilon(u-v)\delta u-\epsilon(u_1-v_1)\delta u_1-\epsilon(u_2-v_2)\delta u_2+[c+v+\epsilon(u-v)]\delta v$$

$$+\epsilon(u_1-v_1)\delta v_1+\epsilon(u_2-v_2)\delta v_2-\frac{\epsilon}{1-\epsilon}\,\frac{b^2}{s}\,\delta s=0$$

$$(c+v)\delta v_1=0\quad,$$

$$(c+v)\delta v_2=0\quad.$$

Therefore, to have non-trivial solutions for the jump quantities either

$$c=-u\quad,\qquad\text{or}\qquad c=-v\quad,\tag{A7}$$

or

$$\begin{vmatrix} c+[\epsilon u+(1-\epsilon)v] & \epsilon\rho & (1-\epsilon)\rho & \frac{\epsilon\rho}{s}\,(u-v) \\ s(c+u) & s\rho & 0 & \rho(c+u) \\ \frac{a^2}{\rho} & c+u+(1-\epsilon)(u-v) & -(1-\epsilon)(u-v) & \frac{b^2}{s} \\ \frac{a^2}{\rho} & -\epsilon(u-v) & c+v+\epsilon(u-v) & -\frac{\epsilon}{1-\epsilon}\,\frac{b^2}{s} \end{vmatrix}=0\quad,$$

or

$$\begin{vmatrix} c+\epsilon u+(1-\epsilon)v & \epsilon & (1-\epsilon) & \epsilon(u-v) \\ c+u & 1 & 0 & c+u \\ a^2 & c+u+(1-\epsilon)(u-v) & -(1-\epsilon)(u-v) & b^2 \\ a^2 & -\epsilon(u-v) & c+v+\epsilon(u-v) & -\frac{\epsilon}{1-\epsilon}\,b^2 \end{vmatrix}=0\quad.$$

The last determinant may be reduced to

$$\begin{vmatrix} c+v & c+(1-\epsilon)u+\epsilon v & 1 \\ a^2-b^2 & (c+u)^2-b^2 & -(u-v) \\ a^2 & (1-\epsilon)(c+u)(c+v)+\epsilon(c+u)^2 & (c+v) \end{vmatrix} = 0 \quad ,$$

which, after expansion, becomes

$$\begin{aligned} &c^4+c^3[2(u+v)]+c^2[u^2+v^2+4uv-a^2-b^2]+c\{2uv(u+v)-2[(1-\epsilon)u+\epsilon v]\,a^2-2[\epsilon u+(1-\epsilon)v]\,b^2\} \\ &+\{u^2v^2-[(1-\epsilon)u^2+\epsilon v^2]\,a^2-[\epsilon u^2+(1-\epsilon)v^2]\,b^2+a^2b^2\} \\ &+(u-v)\{c^3+c^2[(1+\epsilon)u+(2-\epsilon)v]+c[\epsilon u^2+(1-\epsilon)v^2+2uv-a^2]+[\epsilon u^2v+(1-\epsilon)uv^2-a^2(u-\epsilon u+\epsilon v)]\}=0 \end{aligned} \tag{A7}$$

The last term with the factor $(u-v)$ would be absent if the terms $\nabla(\underset{\sim}{v}_n-\underset{\sim}{v}_s)^2$ are absent in Eqs. (5) and (6). Thus the characteristic speeds will be different if these terms are missing. In particular, consider the limiting case $a^2 \to \infty$, i.e. when the liquid can be considered as incompressible, we obtain from (A7):

$$c^2+[2(1-\epsilon)u+2\epsilon v]\,c+[(1-\epsilon)u^2+\epsilon v^2-b^2]+(u-v)\{c+[(1-\epsilon)u+\epsilon v]\} = 0 \quad , \tag{A8}$$

which yields

$$c = \frac{1}{2}\,\{-[(3-2\epsilon)u+(2\epsilon-1)v]\pm[(1-2\epsilon)^2(u-v)^2+4b^2]^{\frac{1}{2}}\} \quad . \tag{A9}$$

On the other hand, when the term with the factor $(u-v)$ in (A7) is missing, we obtain

$$c = -[(1-\epsilon)u+\epsilon v] \pm [b^2-\epsilon(1-\epsilon)(u+v)^2]^{\frac{1}{2}} \quad . \tag{A10}$$

c is always real from (A9), but c may be complex from (A10).

Appendix B

Let us consider the problem of parallel steady flow through a circular pipe with a radius a. In cylindrical coordinates (r, θ, z), we have

$$\underset{\sim}{v}_n = (0, 0, u(r)) \quad , \quad \text{and} \quad \underset{\sim}{v}_s = (0, 0, v(r)) \quad .$$

Then

$$\nabla \cdot \underset{\sim}{v}_n = 0 \quad , \quad \text{and} \quad \nabla \cdot \underset{\sim}{v}_s = 0 \quad ,$$

while

$$\nabla \times \underset{\sim}{v}_n = \left(0 \, , \, -\frac{du}{dr} \, , \, 0\right) \quad \text{and} \quad \nabla \times \underset{\sim}{v}_s = \left(0 \, , \, -\frac{dv}{dr} \, , \, 0\right) \, .$$

Also

$$(\underset{\sim}{v}_n \cdot \nabla)\underset{\sim}{v}_n = 0 \quad \text{and} \quad (\underset{\sim}{v}_s \cdot \nabla)\underset{\sim}{v}_s = 0 \quad .$$

Let us also consider the case of incompressible fluids with T = const. throughout the fluid, then all the physical coefficients may be taken as constants. The equations of motion are now, after neglecting the external force:

$$0 = -\frac{1}{\rho} \nabla p - \frac{1-\epsilon}{2} \nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2 - \frac{1}{\rho_n} \underset{\sim}{F}_n + \frac{1}{\rho_n} \underset{\sim}{F}_{sn} \quad , \tag{B1}$$

and

$$0 = -\frac{1}{\rho} \nabla p + \frac{\epsilon}{2} \nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2 - \frac{1}{\rho_s} \underset{\sim}{F}_s - \frac{1}{\rho_s} \underset{\sim}{F}_{sn} \quad . \tag{B2}$$

Multiply (B1) by ρ_n, and (B2) by ρ_s, and add, we obtain

$$0 = -\nabla p - \underset{\sim}{F}_n - \underset{\sim}{F}_s \quad . \tag{B3}$$

(i) Gorter-Mellink formulation. For this formulation, (B3) becomes

$$\nabla p = -\eta \nabla \times (\nabla \times \underset{\sim}{v}_n) \quad ,$$

whence

$$p = p(z) \quad ,$$

and

$$\frac{dp}{dz} = \eta \frac{1}{r} \frac{d}{dr}\left(r \frac{du}{dr}\right) = -A \quad ,$$

where A is a constant. Then using the boundary condition that u vanishes at $r = a$, we obtain

$$u = \frac{A}{4\eta} (a^2 - r^2) \quad . \tag{B4}$$

Insert (B4) in (B2), we then obtain

$$\alpha\epsilon(v-u)^3 = \frac{A}{\rho} \quad ,$$

or

$$v = u + \left(\frac{A}{\alpha\rho\epsilon}\right)^{\frac{1}{3}} \quad . \tag{B5}$$

(ii) Lin's Formulation. Denote $\eta = \eta^{(nn)} + \eta^{(sn)}$, then (B3) again becomes

$$\nabla p = -\eta \nabla \times (\nabla \times \underset{\sim}{v}_n) \quad ,$$

hence,

$$u = \frac{A}{4\eta} (a^2 - r^2) \quad . \tag{B6}$$

Now, for this formulation, (B2) is

$$0 = -\frac{1}{\rho} \nabla p + \frac{\epsilon}{2} \nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2 - \frac{\eta^{(sn)}}{\rho_s} \nabla \times (\nabla \times \underset{\sim}{v}_n) + \frac{\eta^{(ns)}}{\rho_s} \nabla \times (\nabla \times \underset{\sim}{v}_s)$$
$$- \epsilon(\nabla \times \underset{\sim}{v}_s) \times (\underset{\sim}{v}_n - \underset{\sim}{v}_s) \quad ,$$

which in components form becomes:

$$0 = \frac{A}{\rho} - \frac{\eta^{(sn)}}{\eta} \cdot \frac{A}{\rho_s} - \frac{\eta^{(ns)}}{\rho_s} \frac{1}{r} \frac{d}{dr}\left(r \frac{dv}{dr}\right) \quad , \tag{B7}$$

and

$$0 = \epsilon(u-v) \frac{d}{dr} (u-v) + \epsilon(u-v) \frac{dv}{dr} \quad . \tag{B8}$$

From (B8), we obtain

$$\epsilon (u-v) \frac{du}{dr} = 0 \quad ,$$

or

$$u = v \quad , \tag{B9}$$

which is incompatible with the boundary condition (23).

Also, we obtain from (B7):

$$0 = \frac{1}{\rho} - \frac{1}{\rho_s}\left(\frac{\eta^{(sn)}}{\eta} - \frac{\eta^{(ns)}}{\eta}\right) \quad ,$$

or

$$\frac{\eta^{(nn)}+\eta^{(sn)}}{\rho} = \frac{\eta^{(sn)}-\eta^{(ns)}}{\rho_s} \quad . \tag{B10}$$

If the restriction that T = constant throughout the fluid is relaxed and instead we let $T = T(r)$, then (B8) will be changed to

$$0 = \epsilon(u-v) \frac{d}{dr} (u-v) + s \frac{dT}{dr} + \epsilon(u-v) \frac{dv}{dr} \quad .$$

Hence

$$\epsilon(u-v) \frac{du}{dr} + s \frac{dT}{dr} = 0 \quad , \tag{B11}$$

which determines the radial distribution of temperature. Now from (B7), we obtain

$$v = \frac{A}{4\eta^{(ns)}}\left[(1-\epsilon) - \frac{\eta^{(sn)}}{\eta}\right] r^2 + K \quad . \tag{B12}$$

The boundary condition (23) will lead to,

$$\eta^{(sn)}\frac{du}{dr} - \eta^{(ns)}\frac{dv}{dr} = \beta v^3 \quad , \qquad \text{at} \quad r = a \quad , \tag{B13}$$

whence we obtain

$$v = \frac{A}{4\eta^{(ns)}}\left[\frac{\eta^{(sn)}}{\eta} - (1-\epsilon)\right](a^2 - r^2) - \left[\frac{(1-\epsilon)aA}{2\beta}\right]^{\frac{1}{3}} \quad . \tag{B14}$$

(iii) The HVBK formulation. (B3) for this case becomes

$$0 = -\nabla p - \eta\nabla\times(\nabla\times \underset{\sim}{v}_n) - \lambda(\nabla\times \underset{\sim}{v}_s)\times(\nabla\times \underset{\sim}{\nu}) \quad . \tag{B15}$$

Now

$$\underset{\sim}{\nu} = \frac{\nabla\times \underset{\sim}{v}_s}{|\nabla\times \underset{\sim}{v}_s|} = \left(0, \frac{-\frac{dv}{dr}}{\left|\frac{dv}{dr}\right|}, 0\right) = \left(0, -\operatorname{sgn}\frac{dv}{dr}, 0\right) \quad ,$$

thus

$$\nabla\times \underset{\sim}{\nu} = \left(0,\ 0,\ -\frac{1}{r}\operatorname{sgn}\frac{dv}{dr}\right) \quad .$$

In components, (B15) then becomes:

$$\frac{\partial p}{\partial z} = \eta\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) \quad , \tag{B16}$$

and

$$\frac{\partial p}{\partial r} = -\frac{\lambda}{r}\left|\frac{dv}{dr}\right| \quad . \tag{B17}$$

Thus we have

$$p = -Az + f(r) \quad , \tag{B18}$$

and

$$u = \frac{A}{4\eta} \; (a^2 - r^2) \quad . \tag{B19}$$

For this formulation, (B2) is

$$0 = -\frac{1}{\rho}\nabla p + \frac{\epsilon}{2}\nabla(\underset{\sim}{v}_n - \underset{\sim}{v}_s)^2 - \frac{\lambda}{\rho_s}\underset{\sim}{\omega}\times[\nabla\times\underset{\sim}{\nu}] - \frac{1}{\rho_s}[B_1\underset{\sim}{\omega}\times\underset{\sim}{\xi} + B_2\underset{\sim}{\nu}\times(\underset{\sim}{\omega}\times\underset{\sim}{\xi})$$

$$-B_3\underset{\sim}{\nu}(\underset{\sim}{\omega}\cdot\underset{\sim}{\xi})] \quad . \tag{B20}$$

Now

$$\underset{\sim}{\omega}\times\underset{\sim}{\xi} = \left(-\frac{dv}{dr}[u-v] - \frac{\lambda}{\rho_s}\frac{1}{r}\left|\frac{dv}{dr}\right| , 0, 0\right) ,$$

$$\underset{\sim}{\nu}\times(\underset{\sim}{\omega}\times\underset{\sim}{\xi}) = \left(0, 0, -[u-v]\left|\frac{dv}{dr}\right| - \frac{\lambda}{\rho_s}\frac{1}{r}\frac{dv}{dr}\right)$$

and

$$\underset{\sim}{\omega}\cdot\underset{\sim}{\xi} = 0 \quad .$$

In terms of components, (B20) becomes

$$0 = \frac{A}{\rho} + \frac{B_2}{\rho_s}\left[(u-v)\left|\frac{dv}{dr}\right| + \frac{\lambda}{\rho_s r}\frac{dv}{dr}\right] \quad , \tag{B21}$$

and

$$0 = \left(\frac{1}{\rho} - \frac{1}{\rho_s}\right)\frac{\lambda}{r}\left|\frac{dv}{dr}\right| + \epsilon(u-v)\frac{d}{dr}(u-v) + \frac{B_1}{\rho_s}\left[(u-v)\frac{dv}{dr} + \frac{\lambda}{\rho_s r}\left|\frac{dv}{dr}\right|\right] \quad . \tag{B22}$$

Equations (B21) and (B22) are in general incompatible. Again let us take $T = T(r)$, then (B22) is changed to

$$0 = \left(\frac{1}{\rho} - \frac{1}{\rho_s} + \frac{B_1}{\rho_s^2}\right)\frac{\lambda}{r}\left|\frac{dv}{dr}\right| + \epsilon(u-v)\frac{d}{dr}(u-v) + \frac{B_1}{\rho_s}(u-v)\frac{dv}{dr} + s\frac{dT}{dr} \quad , \tag{B23}$$

which serves to determine the radial distribution of the temperature. For Eq. (B21), we see that $\frac{dv}{dr}$ can only vanish at $r = 0$, and it indeed vanishes there. Hence $\frac{dv}{dr}$ does not change sign. Take A to be positive, we see that $\frac{dv}{dr} \leqslant 0$. Hence (B21) becomes

$$\left[v + \frac{A}{4\eta}(r^2 - a^2) + \frac{\lambda}{\rho_s}\frac{1}{r}\right]\frac{dv}{dr} + \frac{A}{B_2}\frac{\rho_s}{\rho} = 0 \quad . \text{(B24)}$$

Numerical method may be used to integrate the last equation, although series solution can be found in the neighborhood of $r = 0$. However, it is worthwhile to note:

$$\frac{d^2v}{dr^2} = \frac{\frac{A}{B_2}\frac{\rho_s}{\rho}\left(\frac{dv}{dr} + \frac{Ar}{2\eta} - \frac{\lambda}{\rho_s}\frac{1}{r^2}\right)}{\left[v + \frac{A}{4\eta}(r^2 - a^2) + \frac{\lambda}{\rho_s}\frac{1}{r}\right]^2} \xrightarrow{r \to 0} -\frac{A\rho_s^2}{B_2 \lambda \rho} \quad , \tag{B25}$$

and

$$\frac{d^3v}{dr^3} = -\frac{\frac{2A\rho_s}{B_2\rho}\left(\frac{dv}{dr} + \frac{Ar}{2\eta} - \frac{\lambda}{\rho_s}\frac{1}{r^2}\right)^2}{\left[v + \frac{A}{4\eta}(r^2 - a^2) + \frac{\lambda}{\rho_s}\frac{1}{r}\right]^3} + \frac{\frac{A\rho_s}{B_2\rho}\left(\frac{d^2v}{dr^2} + \frac{A}{2\eta} + \frac{2\lambda}{\rho_s}\frac{1}{r^3}\right)}{\left[v + \frac{A}{4\eta}(r^2 - a^2) + \frac{\lambda}{\rho_s}\frac{1}{r}\right]^2}$$

$$\xrightarrow{r \to 0} \frac{2A\rho_s^3}{B_2\rho\lambda^2}\left[v(o) - \frac{Aa^2}{4\eta}\right] \quad . \tag{B26}$$

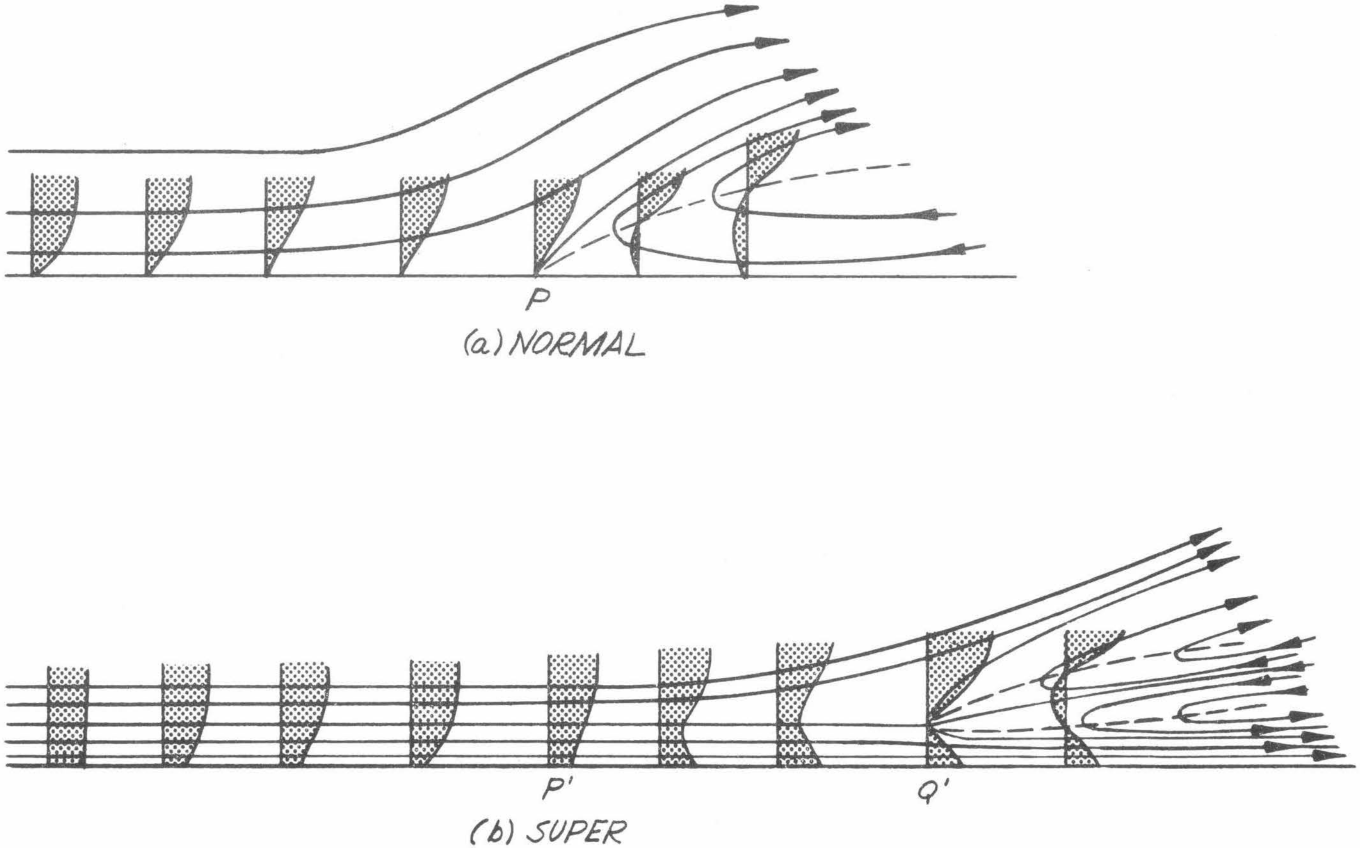

Fig. 1. Velocity profiles for normal (a) and super (b) components, illustrating the boundary-layer separation

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